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Some sharp estimates of the constants of Landau and Lebesgue ☆

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A B S T R A C T

We establish several sharp two-sided inequalities involving the constants of Landau and Lebesgue, which occur and play important roles in two related extremal problems in complex analysis and in Fourier analysis respectively. The main results improves essentially the Watson asymptotic expansion formulas for these constants.

1. Introduction

The constants of Landau and Lebesgue are defined for all integers $n \geqslant 0$ by

$$G_n = \sum_{k=0}^{n} \frac{1}{16^k} \binom{2k}{k}^2 \quad \text{and} \quad L_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin((n+\frac{1}{2})t)}{\sin(\frac{1}{2}t)} \right| dt$$

which occur and play important roles in two related extremal problems in complex analysis and in the theory of Fourier series respectively.

There is a huge literature about the constants of Landau and Lebesgue (interested readers could find useful information in papers [2–8], and the references therein). In particular, H. Alzer [2] proved the following several essentially sharp inequalities for the constants G_n and $L_{n/2}$ in terms of the psi function.

Theorem A. *Let* $c_0 = (1/\pi)(\gamma + 4\log 2) = 1.06627\ldots$. *For all integers* $n \geqslant 0$, *we have*

$$c_0 + \frac{1}{\pi}\Psi(n+\alpha_1) < G_n \leqslant c_0 + \frac{1}{\pi}\Psi(n+\beta_1), \tag{1.1}$$

with the best possible constants

$$\alpha_1 = \frac{5}{4} \quad \textit{and} \quad \beta_1 = \Psi^{-1}\big(\pi(1-c_0)\big) = 1.26621\ldots.$$

Theorem B. *Let*

$$c_1 = \frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{\log k}{4k^2-1} + \frac{4}{\pi^2}(\gamma + 2\log 2) = 0.98943\ldots. \tag{1.2}$$

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For all integers $n \geqslant 0$, we have

$$c_1 + \frac{4}{\pi^2}\Psi(n+a_1) \leqslant L_{n/2} < c_1 + \frac{4}{\pi^2}\Psi(n+b_1), \tag{1.3}$$

with the best possible constants

$$a_1 = \Psi^{-1}\big(\pi^2(1-c_1)/4\big) = 1.48891\ldots \quad \text{and} \quad b_1 = \frac{3}{2}. \tag{1.4}$$

Now by using the following Watson asymptotic formulas

$$G_n = \frac{1}{\pi}\log(n+1) + c_0 - \frac{1}{4\pi(n+1)} + O\left(\frac{1}{n^2}\right) \quad (n \to \infty), \tag{1.5}$$

and

$$L_{n/2} = \frac{4}{\pi^2}\log(n+1) + c_1 + O\left(\frac{1}{n^2}\right) \quad (n \to \infty) \tag{1.6}$$

(see Watson [11]), and

$$\Psi(x) = \log x - \frac{1}{2x} + O\left(\frac{1}{x^2}\right) \quad (x \to \infty) \tag{1.7}$$

(see [1, p. 259]), we find that the error order of the left-hand inequality of (1.1) to G_n is at most $O(n^{-2})$, but the right-hand one is only $O(n^{-1})$. Similarly, the error order of the right-hand inequality of (1.3) to $L_{n/2}$ is just $O(n^{-2})$, but the left-hand one is only $O(n^{-1})$. Thus, the inequalities of (1.1) and (1.3) do not imply the Watson asymptotic formulas (1.5) and (1.6) respectively.

In this paper we will establish sharp inequalities for G_n and $L_{n/2}$ such that imply the above Watson asymptotic formulas respectively. We prove the following.

Theorem 1. *We have for all integers $n \geqslant 1$*

$$\frac{1}{\pi}\log(n+1) + c_0 - \frac{1}{4\pi(n+1)} + \frac{A}{\pi(n+1)^2} < G_n < \frac{1}{\pi}\log(n+1) + c_0 - \frac{1}{4\pi(n+1)} + \frac{A}{\pi(n+1)^2} + \frac{B}{\pi(n+1)^3}, \tag{1.8}$$

where and in the sequel, c_0 is given in Theorem A, *and*

$$A = \frac{5}{192}, \qquad B = \frac{3}{128}.$$

Theorem 2. *We have for all integers $n \geqslant 0$*

$$\frac{4}{\pi^2}\log(n+1) + c_1 + \frac{d_0}{(n+1)^2} - \frac{d_1}{(n+1)^4} < L_{n/2} < \frac{4}{\pi^2}\log(n+1) + c_1 + \frac{d_0}{(n+1)^2} - \frac{d_1}{(n+1)^4} + \frac{d_2}{(n+1)^6}, \tag{1.9}$$

where and in the sequel, c_1 is given in (1.2), *and*

$$d_0 = \frac{12-\pi^2}{18\pi^2} = 0.01199190\ldots,$$
$$d_1 = \frac{7}{120\pi^2}\left(8 - \frac{2\pi^2}{3} - \frac{\pi^4}{90}\right) = 0.00199736\ldots,$$
$$d_2 = \frac{1}{16\pi^2}\left(32 - \frac{8\pi^2}{3} - \frac{2\pi^4}{45} - \frac{\pi^6}{945}\right) = 0.00211774\ldots.$$

Corollary 1. *We have the following new asymptotic formulas for G_n and $L_{n/2}$ when $n \to \infty$,*

$$G_n = \frac{1}{\pi}\log(n+1) + c_0 - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2} + O\left(\frac{1}{(n+1)^3}\right), \tag{1.10}$$

$$L_{n/2} = \frac{4}{\pi^2}\log(n+1) + c_1 + \frac{d_0}{(n+1)^2} - \frac{d_1}{(n+1)^4} + O\left(\frac{1}{(n+1)^6}\right). \tag{1.11}$$

Corollary 2. *We have for all integers $n \geqslant 1$*

$$L_{n/2} - G_n < \frac{4-\pi}{\pi^2}\log(n+1) + (c_1 - c_0) + \frac{1}{4\pi(n+1)} + \frac{e_1}{(n+1)^2} - \frac{d_1}{(n+1)^4} + \frac{d_2}{(n+1)^6}, \tag{1.12}$$

and

$$L_{n/2} - G_n > \frac{4-\pi}{\pi^2}\log(n+1) + (c_1 - c_0) + \frac{1}{4\pi(n+1)} + \frac{e_1}{(n+1)^2} - \frac{e_2}{(n+1)^3} - \frac{d_1}{(n+1)^4}, \tag{1.13}$$

where

$$e_1 = d_0 - \frac{5}{192\pi}, \qquad e_2 = \frac{3}{128\pi}.$$

2. Proofs of theorems

Write

$$P_n = \frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}.$$

The proof of Theorem 1 needs the following lemma (see [12]).

Lemma 1. *For all integers $n \geqslant 1$, it holds*

$$\frac{1}{\sqrt{\pi n\left(1 + \frac{1}{4n(1-\frac{1}{8n+3})}\right)}} < P_n < \frac{1}{\sqrt{\pi n\left(1 + \frac{1}{4n(1-\frac{1}{8n+4})}\right)}}. \tag{2.1}$$

Proof of Theorem 1. Using Lemma 1 for all integers $n \geqslant 1$, we have

$$\frac{32n+8}{32n^2+16n+3} < \pi {P_n}^2 < \frac{8n+3}{8n^2+5n+1}. \tag{2.2}$$

Let

$$x_n = G_n - \frac{1}{\pi}\log(n+1) - c_0 + \frac{1}{4\pi(n+1)} - \frac{\alpha}{\pi(n+1)^2},$$

where α is an undetermined constant. Then using the Watson asymptotic formula (1.5), it holds that

$$\lim_{n\to\infty} x_n = 0. \tag{2.3}$$

We get for $n \geqslant 1$ together with (2.2)

$$\begin{aligned}
\pi(x_n - x_{n-1}) &= \pi {P_n}^2 - \log\left(1+\frac{1}{n}\right) + \frac{1}{4}\left(\frac{1}{n+1} - \frac{1}{n}\right) - \alpha\left(\frac{1}{(n+1)^2} - \frac{1}{n^2}\right)\\
&< \frac{8n+3}{8n^2+5n+1} - \log\left(1+\frac{1}{n}\right) + \frac{1}{4}\left(\frac{1}{n+1} - \frac{1}{n}\right) - \alpha\left(\frac{1}{(n+1)^2} - \frac{1}{n^2}\right)\\
&\overset{\text{def}}{=} f(n).
\end{aligned}$$

Now we consider the function $f(x)$, $x \geqslant 1$, it holds

$$\begin{aligned}
f'(x) &= -\frac{64x^2+48x+7}{(8x^2+5x+1)^2} + \frac{1}{x(x+1)} + \frac{1}{4}\left(\frac{1}{x^2} - \frac{1}{(x+1)^2}\right) - 2\alpha\left(\frac{1}{x^3} - \frac{1}{(x+1)^3}\right)\\
&= -\frac{64x^2+48x+7}{(8x^2+5x+1)^2} + \frac{4x^2(x+1)^2 + x(x+1)(2x+1) - 8\alpha(3x^2+3x+1)}{4x^3(x+1)^3}\\
&\overset{\text{def}}{=} \frac{Q(x)}{4x^3(x+1)^3(8x^2+5x+1)^2},
\end{aligned}$$

where

$$\begin{aligned}
Q(x) &= \left(8x^2+5x+1\right)^2\left[4x^2(x+1)^2 + x(x+1)(2x+1) - 8\alpha\left(3x^2+3x+1\right)\right] - 4x^3(x+1)^3\left(64x^2+48x+7\right)\\
&= 8(5-192\alpha)x^6 + (158-3456\alpha)x^5 + (195-3416\alpha)x^4 + (93-1864\alpha)x^3 + (17-592\alpha)x^2 + (1-104\alpha)x - 8\alpha.
\end{aligned}$$

Now take $\alpha = A = \frac{5}{192}$, we obtain

$$Q(x) = 68x^5 + \frac{2545}{24}x^4 + \frac{1067}{24}x^3 + \frac{19}{12}x^2 - \frac{41}{24}x - \frac{5}{24} > 0, \quad x \geqslant 1.$$

Thus,

$$f'(x) > 0, \quad x \geqslant 1, \quad \text{and} \quad f(\infty) = 0.$$

It implies $f(x) < 0$, $x \geqslant 1$, and then we conclude from (2.3) that $x_n \downarrow 0$, which implies that the left-hand inequality of (1.8) holds.

Similarly, let

$$y_n = G_n - \frac{1}{\pi}\log(n+1) - c_0 + \frac{1}{4\pi(n+1)} - \frac{A}{\pi(n+1)^2} - \frac{B}{\pi(n+1)^3}.$$

We get for $n \geqslant 1$ together with (2.2)

$$\begin{aligned}
\pi(y_n - y_{n-1}) &= \pi P_n{}^2 - \log\left(1 + \frac{1}{n}\right) + \frac{1}{4}\left(\frac{1}{n+1} - \frac{1}{n}\right) - A\left(\frac{1}{(n+1)^2} - \frac{1}{n^2}\right) - B\left(\frac{1}{(n+1)^3} - \frac{1}{n^3}\right) \\
&> \frac{32n+8}{32n^2+16n+3} - \log\left(1 + \frac{1}{n}\right) + \frac{1}{4}\left(\frac{1}{n+1} - \frac{1}{n}\right) - A\left(\frac{1}{(n+1)^2} - \frac{1}{n^2}\right) - B\left(\frac{1}{(n+1)^3} - \frac{1}{n^3}\right) \\
&\stackrel{\text{def}}{=} g(n).
\end{aligned}$$

Now we consider the function $g(x)$, $x > 1$, it holds

$$\begin{aligned}
g'(x) &= -\frac{32^2x^2 + 512x + 32}{(32x^2+16x+3)^2} + \frac{4x^2(x+1)^2 + x(x+1)(2x+1) - 8A(3x^2+3x+1)}{4x^3(x+1)^3} + 3B\frac{(x+1)^4 - x^4}{x^4(x+1)^4} \\
&\stackrel{\text{def}}{=} \frac{R(x)}{4x^4(x+1)^4(32x^2+16x+3)^2},
\end{aligned}$$

where

$$\begin{aligned}
R(x) &= \left(32x^2+16x+3\right)^2\left\{4x^3(x+1)^3 + x^2(x+1)^2(2x+1) - 8Ax(x+1)\left(3x^2+3x+1\right) + 12B\left[(x+1)^4 - x^4\right]\right\} \\
&\quad - 4x^4(x+1)^4\left(32^2x^2 + 512x + 32\right) \\
&= 384(3 - 128B)x^7 + \left(\frac{5198}{3} - 122880B\right)x^6 + \left(\frac{6322}{3} - 144384B\right)x^5 + \left(\frac{14609}{24} - 98304B\right)x^4 \\
&\quad + \left(\frac{281}{12} - 30896B\right)x^3 - \left(\frac{37}{2} + 10632B\right)x^2 - \left(\frac{15}{8} + 1584B\right)x - 108B.
\end{aligned}$$

Now take $B = \frac{3}{128}$, we obtain

$$R(x) = -\frac{3442}{3}x^6 - \frac{3830}{3}x^5 - \frac{40687}{24}x^4 - \frac{16817}{24}x^3 - \frac{4283}{16}x^2 - 39x - \frac{81}{32} \leqslant 0 \quad (n \geqslant 1),$$

together with (2.3), it follows that $y_n \uparrow 0$ which implies that the right-hand inequality of (1.8) holds. $\square$

We need to establish the following inequalities to prove Theorem 2.

Lemma 2. *For $x > 0$, we have*

$$\log x + \frac{1}{24x^2} - \frac{7}{960x^4} < \psi\left(x + \frac{1}{2}\right) < \log x + \frac{1}{24x^2} - \frac{7}{960x^4} + \frac{1}{128x^6}. \tag{2.4}$$

Proof. We first have the following inequalities

$$\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} < \Psi'(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7}. \tag{2.5}$$

In fact, the left-hand inequality of (2.5) is given in [9, Theorem 4]. Using the following

$$\begin{aligned}
\frac{1}{\xi^2(\xi+1)^2} &= \frac{1}{3}\left(\frac{1}{\xi^3} - \frac{1}{(\xi+1)^3}\right) - \frac{1}{15}\left(\frac{1}{\xi^5} - \frac{1}{(\xi+1)^5}\right) + \frac{1}{21}\left(\frac{1}{\xi^7} - \frac{1}{(\xi+1)^7}\right) - \frac{63\xi^4 + 126\xi^3 + 98\xi^2 + 35\xi + 5}{105\xi^7(\xi+1)^7} \\
&< \frac{1}{3}\left(\frac{1}{\xi^3} - \frac{1}{(\xi+1)^3}\right) - \frac{1}{15}\left(\frac{1}{\xi^5} - \frac{1}{(\xi+1)^5}\right) + \frac{1}{21}\left(\frac{1}{\xi^7} - \frac{1}{(\xi+1)^7}\right),
\end{aligned}$$

we obtain the right-hand inequality of (2.5) (see [9, Theorem 4]).

Let

$$f(x)=\Psi\left(x+\frac{1}{2}\right)-\log x-\frac{1}{24x^2}+\frac{7}{960x^4}.$$

Applying the asymptotic expansion (1.7), we get

$$\lim_{x\to\infty} f(x)=0. \tag{2.6}$$

Using the estimates (2.5), we obtain

$$\begin{aligned}
f'(x)&=\Psi'\left(x+\frac{1}{2}\right)-\frac{1}{x}+\frac{1}{12x^3}-\frac{7}{240x^5}\\
&<\frac{1}{x+1/2}+\frac{1}{2(x+1/2)^2}+\frac{1}{6(x+1/2)^3}-\frac{1}{30(x+1/2)^5}+\frac{1}{42(x+1/2)^7}-\frac{1}{x}+\frac{1}{12x^3}-\frac{7}{240x^5}\\
&=-\frac{360x^3+260x^2+70x+7}{7680x^5(x+1/2)^5}+\frac{1}{42(x+1/2)^7}\\
&<0.
\end{aligned}$$

Thus, we have

$$f(x)>\lim_{t\to\infty} f(t)=0 \quad (x>0). \tag{2.7}$$

From (2.7) we conclude that the left-hand inequality of (2.4) holds for $x>0$.

Next, let

$$g(x)=\Psi\left(x+\frac{1}{2}\right)-\left\{\log x+\frac{1}{24x^2}-\frac{7}{960x^4}-\frac{1}{128x^6}\right\}.$$

With the similar way of $f(x)$, we have $g(+\infty)=0$.

Using (2.5), we get

$$\begin{aligned}
g'(x)&=\Psi'\left(x+\frac{1}{2}\right)-\frac{1}{x}+\frac{1}{12x^3}-\frac{7}{240x^5}-\frac{5}{48x^6}\\
&>\frac{1}{x+1/2}+\frac{1}{2(x+1/2)^2}+\frac{1}{6(x+1/2)^3}-\frac{1}{30(x+1/2)^5}-\frac{1}{x}+\frac{1}{12x^3}-\frac{7}{240x^5}+\frac{3}{64x^7}\\
&=-\frac{360x^3+260x^2+70x+7}{7680x^5(x+1/2)^5}+\frac{3}{64x^7}\\
&=\frac{640x^4+830x^3+443x^2+112.5x+45/4}{7680x^7(x+1/2)^5}>0.
\end{aligned}$$

Thus, we have

$$g(x)<\lim_{t\to\infty} g(t)=0 \quad (x>0). \tag{2.8}$$

From (2.8) we conclude that the right-hand inequality of (2.4) holds for $x>0$. □

Proof of Theorem 2. Using the Szegö formula

$$L_{n/2}=\frac{16}{\pi^2}\sum_{k=1}^{\infty}\left(\frac{1}{4k^2-1}\sum_{m=1}^{(n+1)k}\frac{1}{2m-1}\right)$$

(see [10]), and the formula

$$\Psi\left(n+\frac{1}{2}\right)=-\gamma-2\log 2+2\sum_{m=1}^{n}\frac{1}{2m-1}$$

(see [1, p. 258]), we get

$$\begin{aligned}
y_n &\stackrel{\text{def}}{=} L_{n/2}-c_1-\frac{4}{\pi^2}\log(n+1)\\
&=\frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\left(\Psi\left(k(n+1)+\frac{1}{2}\right)-\log k(n+1)\right).
\end{aligned} \tag{2.9}$$

Applying Lemma 2, we have

$$y_n > \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\left(\frac{1}{24k^2(n+1)^2}-\frac{7}{960k^4(n+1)^4}\right)$$
$$= \frac{1}{3\pi^2(n+1)^2}\sum_{k=1}^{\infty}\frac{1}{k^2(4k^2-1)^2}-\frac{7}{120\pi^2(n+1)^4}\sum_{k=1}^{\infty}\frac{1}{k^4(4k^2-1)}. \tag{2.10}$$

Using the following formulas

$$\sum_{k=1}^{\infty}\frac{1}{4k^2-1}=\frac{1}{2},\qquad \sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6},\qquad \sum_{k=1}^{\infty}\frac{1}{k^4}=\frac{\pi^4}{90},\qquad \sum_{k=1}^{\infty}\frac{1}{k^6}=\frac{\pi^6}{945},$$

we get

$$\sum_{k=1}^{\infty}\frac{1}{k^2(4k^2-1)}=4\sum_{k=1}^{\infty}\left(\frac{1}{4k^2-1}-\frac{1}{4k^2}\right)=2-\frac{\pi^2}{6}, \tag{2.11}$$
$$\sum_{k=1}^{\infty}\frac{1}{k^4(4k^2-1)}=4\sum_{k=1}^{\infty}\left(\frac{1}{k^2(4k^2-1)}-\frac{1}{4k^4}\right)=8-\frac{2\pi^2}{3}-\frac{\pi^4}{90}, \tag{2.12}$$

and

$$\sum_{k=1}^{\infty}\frac{1}{k^6(4k^2-1)}=4\sum_{k=1}^{\infty}\left(\frac{1}{k^4(4k^2-1)}-\frac{1}{4k^6}\right)=32-\frac{8\pi^2}{3}-\frac{2\pi^4}{45}-\frac{\pi^6}{945} \tag{2.13}$$

for all $n \geqslant 0$, then we obtain from (2.10)–(2.12):

$$y_n > \frac{1}{3\pi^2(n+1)^2}\left(2-\frac{\pi^2}{6}\right)-\frac{7}{120\pi^2(n+1)^4}\left(8-\frac{2\pi^2}{3}-\frac{\pi^4}{90}\right)$$
$$= \frac{d_0}{(n+1)^2}-\frac{d_1}{(n+1)^4}, \tag{2.14}$$

which implies the left-hand inequality of (1.9) holds.

Similarly, applying Lemma 2, (2.11)–(2.13), we get

$$y_n < \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\left(\frac{1}{24k^2(n+1)^2}-\frac{7}{960k^4(n+1)^4}+\frac{1}{128k^6(n+1)^6}\right)$$
$$= \frac{d_0}{(n+1)^2}-\frac{d_1}{(n+1)^4}+\frac{1}{16\pi^2(n+1)^6}\sum_{k=1}^{\infty}\frac{1}{k^6(4k^2-1)}$$
$$= \frac{d_0}{(n+1)^2}-\frac{d_1}{(n+1)^4}+\frac{d_2}{(n+1)^6}. \tag{2.15}$$

This finishes the proof. □

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